# *Systèmes entrée-sortie non linéaires et applications en audio-acoustique*

## **Séries de Volterra**

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2019

# Plan

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1. Préambule
2. Séries de Volterra : généralités
3. Calcul des noyaux de Volterra d'un système différentiel
4. Exercices et applications en audio-acoustique
5. Convergence
6. Extension en dimension infinie et application
7. Conclusion

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# Outline

# Recall: standard result (1/2) (see e.g. [Boyd,1984])

## Recall: definition of a Volterra series

$$x(t) = \sum_{n=1}^{+\infty} x_n(t) \textbf{ with } x_n(t) = \int_{\mathbb{R}^n} h_n(\tau_1, \ldots, \tau_n)\, u(t-\tau_1) \ldots u(t-\tau_n)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n$$

## Bounded Input Bounded Output (BIBO) result

$\left(\|u\|_\infty = \sup_{t\in\mathbb{R}} \left|u(t)\right|\right)$

$$\begin{aligned}
|x_n(t)| &= \left| \int_{\mathbb{R}^n} h_n(\tau_1, \ldots, \tau_n)\, u(t-\tau_1) \ldots u(t-\tau_n)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n \right| \\
&\le \int_{\mathbb{R}^n} \left| h_n(\tau_1, \ldots, \tau_n) \right| \underbrace{\left| u(t-\tau_1) \right| \ldots \left| u(t-\tau_n) \right|}_{\le \|u\|_\infty \ldots \|u\|_\infty} \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n \\
\|x_n\|_\infty &\le \underbrace{\int_{\mathbb{R}^n} \left| h_n(\tau_1, \ldots, \tau_n) \right| \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n}_{=\|h_n\|_1 \quad (L^1\text{-norm})} \left( \|u\|_\infty \right)^n
\end{aligned}$$

**Hence,** $\|x\|_\infty \le \sum_{n=1}^{+\infty} \|x_n\|_\infty \le \sum_{n=1}^{+\infty} \|h_n\|_1 \left( \|u\|_\infty \right)^n.$

# Recall: standard result (2/2) (see e.g. [Boyd,1984])

**Gain bound function $\varphi$**

Define $\varphi(z) = \sum_{n\geq 1} \|h_n\|_1 z^n$ with convergence radius $\rho$ at $z = 0$.

**Theorem (BIBO result)**

**If** $\|u\|_\infty < \rho$, **then** the Volterra series expansion of $x$ is normally convergent and

$$\|x\|_\infty \leq \varphi\left(\|u\|_\infty\right) < +\infty.$$

Moreover, the **truncation error is bounded**:

$$\left\| \sum_{n=N+1}^{+\infty} x_n \right\|_\infty \leq \sum_{n=N+1}^{+\infty} \|h_n\|_1 \left(\|u\|_\infty\right)^n$$

**QUESTION**

Can we use these theoretical results in practice?

# Application test on $\dot{x} + ax - \varepsilon x^3 = u$ $\quad (a > 0,\ \varepsilon > 0)$

**Using interconnection laws in the Laplace domain:**

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## Using interconnection laws in the Laplace domain:

$(\widehat{s_{1:n} + a}) H_n(s_{1:n})$

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**(Formula with convolutions are also available in the time-domain)**

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**(Formula with convolutions are also available in the time-domain)**

## Even for this basic example...

Computing $\|h_n\|_1$ is difficult in practice because of the rapidly increasing number of terms !

# Outline

# Functional setting & Systems ($\mathbb{T} = \mathbb{R}_+$ or $[0, T]$ with $T > 0$)

(input) $\mathscr{U} = L^\infty(\mathbb{T}, \mathbb{U})$ with $\mathbb{U} = \mathbb{R}^q$ *(or a Banach space)*
(state) $\mathscr{X} = L^\infty(\mathbb{T}, \mathbb{X})$ with $\mathbb{X} = \mathbb{R}^p$ *(idem)*

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## Operators spaces

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  with norm $\|E\| = \sup_{\substack{(x_1,\ldots,x_m) \in \mathbb{X}^m \\ \|x_1\| = \cdots = \|x_m\| = 1}} \|E(x_1, \ldots, x_k)\|_{\mathbb{X}}$

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## Class of systems

$$\dot{x} = Ax + Bu + \sum_{m=2}^{M} A_m(\underbrace{x, \ldots, x}_{m}), \text{ for } t \in \mathbb{T} \text{ with } x(0) = 0$$

$A \in \mathscr{L}(\mathbb{X})$ ($\equiv p \times p$ matrix), $B \in \mathscr{L}(\mathbb{U}, \mathbb{X})$ ($\equiv p \times q$), $A_m \in \mathscr{M}\mathscr{L}_m(\mathbb{X})$

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## Example ($\dot{x} = -ax + u + \varepsilon x^3$, $a>0$, $\varepsilon>0$)

$A=-a$, $B=1$, $A_3(x_1,x_2,x_3)=\varepsilon\, x_1\, x_2\, x_3$ and $\|A_3\|_{\mathscr{M}\mathscr{L}}=\varepsilon$

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**Rk:** Solution of $\dot{x} = Ax + v$, $x(0) = 0$: $x(t) = \int_0^t S(\tau) v(t - \tau) \mathrm{d}\tau$
with $S(t) = e^{At} 1_{\mathbb{R}_+}(t)$ *(or $C^0$ semi-group [Curtain,Zwart,1995])*

| | | |
|---|---|---|
| Hyp.: | (i) | $\exists \alpha \in \mathbb{R}, \beta > 0$ s.t. $\forall t \in \mathbb{T}$, $\|S(t)\|_{\mathscr{L}(\mathbb{X})} \leq \beta e^{\alpha t}$ |
| | (ii) | if $\mathbb{T} = \mathbb{R}_+$, $\alpha < 0$ (matrix $A$ is Hurwitz) |

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$A = -a$, $B = 1$, $A_3(x_1, x_2, x_3) = \varepsilon\, x_1\, x_2\, x_3$ and $\|A_3\|_{\mathscr{M}\mathscr{L}} = \varepsilon$
$S(t) = h_1(t) = e^{-at} 1_{\mathbb{R}_+}(t)$, $\alpha = -a < 0$ and $\beta = 1$

# (Back to the) time-domain Volterra series expansion

## Applying the perturbation method *(recall the preambule)*

The series expansion of the trajectories is $x(t) = \sum_{n=1}^{+\infty} x_n(t)$

where $x_n$ is solution of $\dot{x}_n = A x_n + \chi_n, \quad x_n(0) = 0$

with $\chi_1(t) = B u(t)$ and $\chi_{n\geq 2}(t) = \sum_{k=2}^{\max(M,n)} \sum_{p\in\mathbb{M}_n^k} A_k\left(x_{p_1}(t),\ldots,x_{p_k}(t)\right)$

and $\mathbb{M}_n^K = \left\{p \in (\mathbb{N}^*)^K \mid p_1 + \cdots + p_K = n\right\}$

## Solution: convolution by the semi-group $S$

For all $n \geq 1$ and $t \in \mathbb{T}$, $x_n(t) = \int_0^t S(\tau)\chi_n(t-\tau)\,\mathrm{d}\tau$

## Example ($\dot{x} = -ax + u + \varepsilon x^3, \quad a > 0,\ \varepsilon > 0$)

$x_1(t) = \int_0^t e^{-a\tau}\, u(t-\tau)\,\mathrm{d}\tau$ *(even oders are zero)*

$x_{n\geq 3}(t) = \int_0^t e^{-a\tau}\left(\sum_{p\in\mathbb{M}_n^3} \varepsilon\, x_{p_1}(t-\tau)\, x_{p_2}(t-\tau)\, x_{p_3}(t-\tau)\right)\mathrm{d}\tau$

# Convergence theorem [Hélie,Laroche]

**Definition (function $F$)**

Let $\gamma = \int_{\mathbb{T}} \|S(t)\|_{\mathscr{L}(\mathbb{X})} \mathrm{d}t$ $\quad (= \|S\|_{\mathbb{S}}$ with $\mathbb{S} = L^1(\mathbb{T}, \mathscr{L}(\mathbb{X})))$

Define $F(X) = \dfrac{X}{X - \gamma Q(X)}$ where $Q(X) = \sum_{m=2}^{M} \|A_m\|_{\mathscr{M}\mathscr{L}} X^m$

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**Theorem 1 (convergence domain and bound)**

(i) There exists a unique $\sigma > 0$ s.t. $F(\sigma) - \sigma F'(\sigma) = 0$

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(iv) If $\|x_1\|_{\mathscr{X}} < \rho^\star$, the series $x = \sum_{n\geq 1} x_n$ converges in norm in $\mathscr{X}$ and $\|x\|_{\mathscr{X}} \leq \Psi\big(\|x_1\|_{\mathscr{X}}\big) < +\infty$

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## Corollary

Denote $v = \|t \mapsto S(t)B\|_{L^1(\mathbb{T}, \mathscr{L}(\mathbb{U}, \mathbb{X}))}$

Let $\rho_u = \rho^\star / v$. If $\|u\|_{\mathscr{U}} < \rho_u$, then $\|x_1\|_{\mathscr{X}} \leq v\|u\|_{\mathscr{U}} < \rho^\star$ and $\|x\|_{\mathscr{X}} \leq \Psi\big(\|x_1\|_{\mathscr{X}}\big) \leq \Psi\big(v\|u\|_{\mathscr{U}}\big)$.

# Proof of the theorem (1/2)

- Given $F$, (i-iii) stem from the singular inversion theorem.
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- Why this $F$? Proof of (iv)? $\rightarrow$Sketch of proof

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Let $\psi_1 = 1$ and $\psi_{n\geq 2} = \gamma \sum_{k=2}^{\max(M,n)} \left[ \|A_k\|_{\mathscr{M}\mathscr{L}} \sum_{p \in \mathbb{M}_n^k} \prod_{i=1}^{k} \psi_{p_i} \right]$.

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We have

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We have by induction

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- Given $F$, (i-iii) stem from the singular inversion theorem.
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$$= \Psi(X) - X$$

leading to $\Psi(X) = X\, F(\Psi(X))$ (→ use the singular inversion theorem) □

# Truncation error bound

**Theorem 2 (truncation error bound)**

Introduce the truncated series $T_N x = \sum_{n=1}^{N} x_n$ and $T_N \Psi(X) = \sum_{n=1}^{N} \psi_n X^n$.

If $\|x_1\|_{\mathscr{X}} < \rho^\star$ (or $\|u\|_{\mathscr{U}} < \rho_u$), then

$$\|x - T_N x\|_{\mathscr{X}} \leq \left[\Psi - T_N \Psi\right]\left(\|x_1\|_{\mathscr{X}}\right) \ \left(\text{or } \leq \left[\Psi - T_N \Psi\right]\left(\nu \|u\|_{\mathscr{U}}\right)\right)$$

The proof stems from step 1

# Back to the example

**Example ($\dot{x} = -ax + u + \varepsilon x^3, \quad a > 0,\ \varepsilon > 0$)**

$A = -a$, $B = 1$, $A_3(x_1, x_2, x_3) = \varepsilon\, x_1\, x_2\, x_3$ and $\|A_3\|_{\mathscr{ML}} = \varepsilon$
$S(t) = h_1(t) = e^{-at} 1_{\mathbb{R}_+}(t)$, $\alpha = -a < 0$ and $\beta = 1$

$$\gamma = \int_0^{+\infty} e^{-at}\, \mathrm{d}t = \frac{1}{a} \quad (= \nu)$$

$$F(X) = \frac{X}{X - \gamma \|A_3\| X^3} = \frac{1}{1 - (\varepsilon/a) X^2}$$

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$$A = -a,\ B = 1,\ A_3(x_1, x_2, x_3) = \varepsilon\, x_1\, x_2\, x_3 \quad \text{and} \quad \|A_3\|_{\mathscr{ML}} = \varepsilon$$
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(i) $\sigma > 0$ s.t. $F(\sigma) - \sigma F'(\sigma) = 0$: $1 - \dfrac{3\varepsilon}{a}\sigma^2 = 0$ and $\sigma = \sqrt{\dfrac{a}{3\varepsilon}}$

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(iii) $\rho^\star = \dfrac{\sigma}{F(\sigma)} > 0$: $\rho^\star = \underbrace{\left(1 - \dfrac{\varepsilon}{a}\sigma^2\right)}_{2/3}\sigma$ $\qquad \rho^\star = \dfrac{2}{3}\sqrt{\dfrac{a}{3\varepsilon}}$,

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(i) $\sigma > 0$ s.t. $F(\sigma) - \sigma F'(\sigma) = 0$: $1 - \frac{3\varepsilon}{a}\sigma^2 = 0$ and $\sigma = \sqrt{\frac{a}{3\varepsilon}}$

(ii) $\Psi(z) = z\, F\big(\Psi(z)\big)$ analytic at $z = 0$: $\boxed{\Psi(z) - \frac{\varepsilon}{a}\Psi(z)^3 = z}$

(iii) $\rho^\star = \frac{\sigma}{F(\sigma)} > 0$: $\rho^\star = \underbrace{\left(1 - \frac{\varepsilon}{a}\sigma^2\right)}_{2/3}\sigma$ $\qquad \rho^\star = \frac{2}{3}\sqrt{\frac{a}{3\varepsilon}}$,

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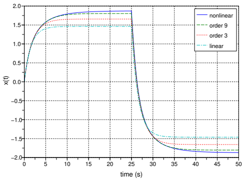
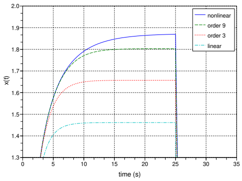
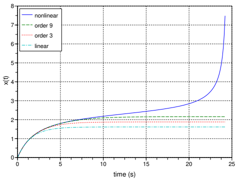
**Remark**

When there is no closed-form solution for $\gamma$, $\sigma$, $\rho^\star$ (...), equations of steps (i-iii) can be numerically solved.

# Numerical simulations

($a = 0.65$ and $\varepsilon$ s.t. $\rho_u = 1$)

## Square input: $u(t) = e$ on $[0, 25)$ and $u(t) = -e$ on $[25, 50)$, etc.

| $e = 0.95 < \rho_u$ | Idem: ZOOM | $e = 1.05 > \rho_u$ |
|---|---|---|
|   |   |   |

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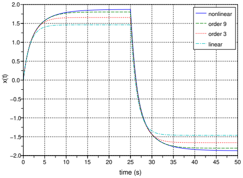
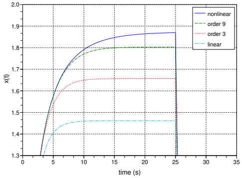
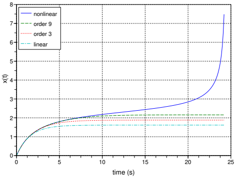
|  |  |  |



- $e < \rho^{\star}$: the VS converges to the trajectory of the NL system
- $e > \rho^{\star}$: the VS becomes divergent and the trajectory of the nonlinear system gets out of the domain of attraction of 0.

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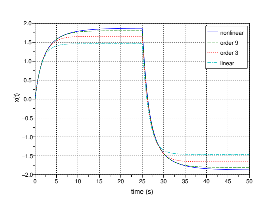
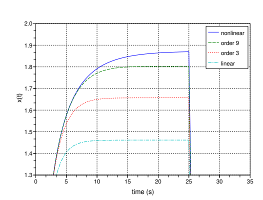
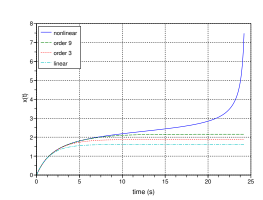
| $e = 0.95 < \rho_u$ | Idem: ZOOM | $e = 1.05 > \rho_u$ |
|---|---|---|
|  |  |  |



- $e < \rho^\star$: the VS converges to the trajectory of the NL system
- $e > \rho^\star$: the VS becomes divergent and the trajectory of the nonlinear system gets out of the domain of attraction of 0.

In this example (not in general!): (a) $\rho^\star$ and $\rho_u$ are tight bounds; (b) the convergence domain coincides with the domain of attraction of 0.

# Generalizations of theorems 1 and 2

**[IEEE-TAC 2011]** (finite-dim. systems)

- Affine systems: $\dot{x} = Ax + Bu + P(x) + \mathrm{Bilin}(Q(x), u)$
  with $P(x) = \sum_{m=2}^{+\infty} A_m(x, \ldots, x)$
  and $Q(x) = \sum_{m=1}^{+\infty} B_m(x, \ldots, x)$
- $\mathscr{U}, \mathscr{X}$: Weighted-$L^\infty$ spaces
  (exponentially-damped input/output results)

**[Automatica 2014]** (infinite-dim. systems)

- $\mathbb{X}$, $\mathbb{U}$: Banach spaces on the field $\mathbb{R}$
- $\dot{x} = L(x, u) + P(x) + \mathrm{Bilin}(Q(x), u)$ where
  $\dot{x} = L(x, u)$ defines a distributed or boundary control linear system
- Non zero initial conditions

# Plan

# Outline

# Recall (§2): Volterra kernels of time-Varying systems

## A definition is also available for time-varying systems:

$$y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} g_n(t, \theta_1, \ldots, \theta_n)\, u(\theta_1) \ldots u(\theta_n)\, \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n$$

## Time-Invariant (TI) case and link with kernels $h_n$

TI case: $y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} h_n(\tau_1, \ldots, \tau_n)\, u(t-\tau_1) \ldots u(t-\tau_n)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n$

Kernels $g_n$ of a TI system are such that ($\theta_k = t - \tau_k$)

$$g_n(t, t-\tau_1, \ldots, t-\tau_n) = h_n(\tau_1, \ldots, \tau_n)$$

does not depend on $t$

## Causal kernels $g_n$

$\exists k$ s.t. $\theta_k > t \Rightarrow g_n(t, \theta_1, \ldots, \theta_k, \ldots, \theta_n) = 0$

# Green-Volterra kernels

Basic idea: space-time problem $\to$ space-time kernels.

**Definition** $\xrightarrow{u(r,t)} \boxed{\{g_n\}} \xrightarrow{y(r,t)}$ $\qquad (r,t) \in \Omega \times \mathbb{R}$

A system is defined by the Green-Volterra series $\{g_n\}_{n \geq 1}$ if

$$u(r,t) = \sum_{n=1}^{+\infty} \int_{(\Omega\times\mathbb{R})^n} g_n(r,t;\rho_1,\theta_1,\ldots,\rho_n,\theta_n)\, u(\rho_1,\theta_1)\ldots u(\rho_n,\theta_n) \times \mathrm{d}\rho_1\,\mathrm{d}\theta_1\ldots\mathrm{d}\rho_n\,\mathrm{d}\theta_n$$

**Time-invariant kernels and Transfer kernels**

- Time-invariance : $\exists h_n$ s.t. $\forall r,t$ and $\rho_1,\ldots,\rho_n$ and $\tau_1,\ldots,\tau_n$,

$$g_n(r,t;\rho_1,t-\tau_1,\ldots,\rho_n,t-\tau_n) = h_n(r;\rho_1,\tau_1,\ldots,\rho_n,\tau_n)$$

- Transfer kernels = Laplace transform of $h_n$ w.r.t. to $\tau_1,\ldots,\tau_n$

$$= H_n(r;\rho_1,s_1,\ldots,\rho_n,s_n)$$

$\to$ Interconnection laws can be generalized

# A damped Kirchhoff-Carrier String [Roze,Hélie,JSV2014]

**Dimensionless model** $(\Omega = ]0,1[,\ \mathbb{T} = \mathbb{R}_+)$

$$\underbrace{\partial_t^2 w + 2\alpha\partial_t w - \partial_x^2 w}_{L_{x,t}[w]} + \underbrace{\varepsilon\left[\int_0^1 \left(\partial_x w\right)^2 \mathrm{d}x\right]\partial_x^2 w}_{-A_3(w,w,w)} = f \quad \text{in } \Omega \times \mathbb{T}$$

(BC: Dirichlet ; IC: zero ; damping: $\alpha > 0$; NL: $\varepsilon > 0$)

# A damped Kirchhoff-Carrier String [Roze,Hélie,JSV2014]

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(BC: Dirichlet ; IC: zero ; damping: $\alpha > 0$; NL: $\varepsilon > 0$)

**Linearized problem: Green function** $(\mathrm{Re}(s) > -2\alpha)$

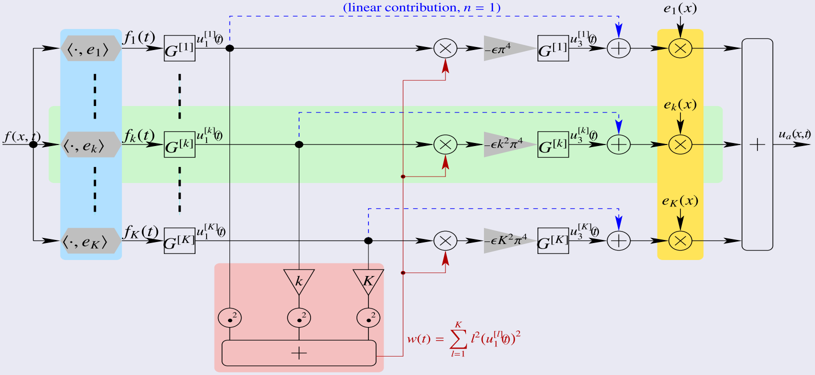
| Space-Time | Space-Laplace |
|---|---|
| $L_{x,t}[g_1](x,t;\xi,\theta) = \delta(x-\xi)\otimes\delta(t-\theta)$ | $[\underbrace{s^2 + 2\alpha s}_{\Gamma(s)^2} - \partial_x^2]H_1(x;\xi,s) = \delta(\xi - x)$ |

$$H_1(x;s,\xi) = \frac{\cosh\left((-1+x+\xi)\Gamma(s)\right) - \cosh\left((1-|x-\xi|)\Gamma(s)\right)}{2\Gamma(s)\sinh\left(\Gamma(s)\right)}$$

## Higher order Green-Volterra kernels (details in the paper)

$$H_n(x;\xi_1,s_1,\dots,\xi_n,s_n) = \int_\Omega H_1(x;\zeta,s_1+\cdots+s_n)$$
$$\times \sum_{p\in\mathbb{M}_n^3} A_\zeta^3\Big(H_{p_1}(\zeta;\xi_1,s_1,\dots),H_{p_2}(\zeta;\xi_{p_1+1},s_{p_1+1},\dots),H_{p_3}(\zeta;\dots)\Big)\,\mathrm{d}\zeta$$

## Decomposition on eigenfunctions $e_k$ + Order reduction



# Outline

1. Préambule
2. Séries de Volterra : généralités
3. Calcul des noyaux de Volterra d'un système différentiel
4. Exercices et applications en audio-acoustique
5. Convergence
6. Extension en dimension infinie et application
   - Noyaux de Green-Volterra : principe et exemple
   - *Bonus: nonlinear beam with convergence results [ICSV-2019]*
7. Conclusion

# Simulation of a damped nonlinear beam

*based on modal decomposition and Volterra series*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$26^{th}$ International Congress on Sound and Vibration
10 July, 2019
Montéal, Canada

## Euler-Bernoulli Model

**Damped nonlinear pinned beam excited by a distributed force with:**

(H1) Euler-Bernoulli kinematics

*(any cross-section before deformation remains straight after deformation)*

(H2) Viscous ($a$) and structural ($b$) damping phenomena

(H3) Von Karman's assumptions

*(coupling between axial and bending movements → nonlinearity )*

**Dimensionless PDE: deflection waves $w(z,t)$** $\quad (z \in \Omega = ]0,1[, \quad t \in \mathbb{T} = \mathbb{R}_+)$

$$\partial_t^2 w + 2\left(a + b\partial_z^4\right)\partial_t w + \partial_z^4 w - \eta \left( \int_0^1 \left(\partial_z w\right)^2 \mathrm{d}z \right) \partial_z^2 w = f \quad \text{(distributed force)}$$

**Conditions and coefficients**

-Pinned beam: $w(z,t) = 0$ (fixed) and $\partial_z^2 w(z,t) = 0$ (no momentum) at $z \in \{0,1\}$

-Zero initial conditions: $w(z, t=0) = 0$ and $\partial_t w(z, t=0) = 0$

-Damping: $a > 0$, $b > 0$

-Nonlinear coupling: $\eta > 0$

# Functional setting

$$\partial_t^2 w + \partial_z^4 w + 2\left(a + b\partial_z^4\right)\partial_t w - \eta\left(\int_0^1 (\partial_z w)^2 \mathrm{d}z\right)\partial_z^2 w = f$$ +BC+IC

## State-Space Formulation for $u \in \mathscr{U}$ with $\mathbb{U} = \mathbb{H} := L^2(0,1)$

$$\dot{x} = Ax + Bu + A_3(x,x,x)$$

with $x = [w, \dot{w}]^T$ and $u = f$

$$Ax = \begin{bmatrix} 0 & I \\ -\Delta^2 & -2(aI + b\Delta^2) \end{bmatrix} x,\ Bu = \begin{bmatrix} 0 \\ u \end{bmatrix},\ A_3(a,b,c) = \begin{bmatrix} 0 \\ -\eta\left(\int_0^1 a_1'(z) b_1'(z)\,\mathrm{d}z\right) c_1'' \end{bmatrix}$$

# Functional setting

$$\partial_t^2 w + \partial_z^4 w + 2(a + b\partial_z^4)\partial_t w - \eta\left(\int_0^1 (\partial_z w)^2 \mathrm{d}z\right)\partial_z^2 w = f$$ +BC+IC

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## Bi-laplacian and laplacian op.: domains and properties

(unbounded in $\mathbb{H}$) [Curtain, Jacob, Zwart, ...]

$\Delta^2$: $D(\Delta^2) = \{ w \in H^4(0,1) \text{ s.t. } w(0) = w(1) = 0,\ w''(0) = w''(1) = 0 \}$

$\Delta^2$ is closed, densely defined, self-adjoint, positive on $\mathbb{H}$

$\Rightarrow$ $\exists$ uniquely defined positive square root $(-\Delta)$

$\Delta$: $D(\Delta) = \{ w \in H^2(0,1) \text{ s.t. } w(0) = w(1) = 0 \}$

$\mathbb{H}^{\frac{1}{2}} = D(\Delta)$ endowed with norm $\|\cdot\| = \|\Delta\ \cdot\|_{\mathbb{H}}$ is a Hilbert space

# Functional setting

$$\boxed{\partial_t^2 w + \partial_z^4 w + 2(a + b\partial_z^4)\partial_t w - \eta\left(\int_0^1 (\partial_z w)^2 \mathrm{d}z\right)\partial_z^2 w = f}$$ +BC+IC

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## Well-posedness: linearized system [Curtain, Jacob, Zwart, ...]

- $\mathbb{U} = \mathbb{H}$ and $\mathbb{X} = \mathbb{H}^{\frac{1}{2}} \times \mathbb{H}$ with norm $\|x\|_{\mathbb{X}} = \left(\|x_1\|^2_{\mathbb{H}^{\frac{1}{2}}} + \|x_2\|^2_{\mathbb{H}}\right)^{\frac{1}{2}}$
- $A$ on $D(A) = \{(u,v) \in \mathbb{H}^{\frac{1}{2}} \times \mathbb{H}^{\frac{1}{2}}, 2(a + b\partial_z^4)v + \partial_z^4 u \in \mathbb{H}\}$ generates a $C^0$ contraction semigroup on $\mathbb{X}$ $\Rightarrow \gamma = \|S\| < \infty$
- $B \in \mathscr{L}(\mathbb{U}, \mathbb{X})$ and $\|B\|_{\mathscr{L}} = 1$.

# Functional setting

$$\partial_t^2 w + \partial_z^4 w + 2(a + b\partial_z^4)\partial_t w - \eta\left(\int_0^1 (\partial_z w)^2 \mathrm{d}z\right)\partial_z^2 w = f$$ +BC+IC

## State-Space Formulation for $u \in \mathscr{U}$ with $\mathbb{U} = \mathbb{H} := L^2(0,1)$

$$\dot{x} = Ax + Bu + A_3(x,x,x)$$ with $x = [w, \dot{w}]^T$ and $u = f$

$$Ax = \begin{bmatrix} 0 & I \\ -\Delta^2 & -2(aI + b\Delta^2) \end{bmatrix} x,\ Bu = \begin{bmatrix} 0 \\ u \end{bmatrix},\ A_3(a,b,c) = \begin{bmatrix} 0 \\ -\eta\left(\int_0^1 a_1'(z) b_1'(z) \mathrm{d}z\right) c_1'' \end{bmatrix}$$

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- $B \in \mathscr{L}(\mathbb{U}, \mathbb{X})$ and $\|B\|_{\mathscr{L}} = 1$.

## Well-posedness: nonlinear system

$$\|A_3\|_{\mathscr{M}\mathscr{L}_3(\mathbb{X},\mathbb{X},\mathbb{X})} < a_3 := \eta/(3\sqrt{10})$$

$\Rightarrow \exists \rho^\star, \rho_u > 0$ (Validity domain $\|u\| < \rho_u$)

## Modal decomposition (linearized model: $\partial_t^2 w + 2\left(a + b\partial_z^4\right)\partial_t w + \partial_z^4 w = f$)

**Basis of $L^2(0,1)$: $\{e_m\}_{m\geq 1}$ with $e_m(z) = \sqrt{2}\sin(k_m z)$ for $k_m = m\pi$**

Eigenfunctions: $\partial_z^4 e_m = k_m^4 e_m$ and BC satisfied

Normed and orthogonal: $\|e_m\|_{L^2} = 1$ and $< e_m, e_n >_{L^2} = 0$ if $m \neq n$

**Model order reduction on $M$ modes: $E = [e_1, \ldots, e_M]^T$**

Excitation: $f(z,t) = E(z)^T F(t)$ with $F = [f_1, \ldots, f_M]^T$

Decomposition (exact): $w(z,t) = E(z)^T W(t)$ with $W = [w_1, \ldots, w_M]^T$

Dynamics of modes $(1 \leq m \leq M)$ : $\ddot{w}_m + 2\left(a + bk_m^4\right)\dot{w}_m + k_m^4 w_m = f_m$

**State-space representation (dimension$= 2M$)**

Input $u = F$, state $x = [W^T, \dot{W}^T]^T$

Dynamic equation:
$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$x(0) = 0_{2M\times 1}.$$

$$A = \begin{pmatrix} 0_{M\times M} & I_M \\ -K^4 & -2(a\,I_M + b\,K^4) \end{pmatrix}, \quad B = \begin{pmatrix} 0_{M\times M} \\ I_M \end{pmatrix} \quad \text{and} \quad K = \mathrm{diag}(k_1, \ldots, k_M)$$

## Solution and properties

$(u \in \mathcal{U} = L^\infty(\mathbb{T}, \mathbb{U})$ living in $\mathbb{U} = \mathbb{R}^M$
and $x_{\text{lin}} \in \mathcal{X} = L^\infty(\mathbb{T}, \mathbb{X})$ living in $\mathbb{X} = \mathbb{R}^{2M})$

**Solution $u \to \boxed{S} \to x_{\text{lin}}$**

$$x_{\text{lin}}(t) = \int_0^\infty S(t-\tau)Bu(\tau)\,\mathrm{d}\tau \quad \text{with} \quad S(t) = \exp(A\,t)1_{t>0}.$$

**Bounded input $(u \in \mathcal{U}) \Rightarrow$ bounded output $(x \in \mathcal{X})$. Why ?**

Due to damping, $\|S(t)\|_{\mathcal{L}(\mathbb{X},\mathbb{X})}$ is exponentially decreasing so that

$$\|x_{\text{lin}}\|_{\mathcal{X}} \leq \gamma \underbrace{\|B\|_{\mathcal{L}(\mathbb{X},\mathbb{U})}}_{1} \|u\|_{\mathcal{U}} \quad \text{where} \quad \gamma \geq \gamma^\star := \int_{\mathbb{T}} \|S(t)\|_{\mathcal{L}(\mathbb{X},\mathbb{X})}\,\mathrm{d}t > 0$$

**Hypotheses for musical applications**

(i) the first mode $(m = 1)$ has a damped oscillating dynamics $(\frac{a}{\pi^2} + b\pi^2 < 1)$

(ii) higher modes are more damped $(2b(a - b\pi^4) \leq 1)$

$\Rightarrow \gamma = 1/(a + b\pi^4)$ is a bound

Nonlinear problem $\left(\partial_t^2 w + 2\left(a + b\partial_z^4\right)\partial_t w + \partial_z^4 w - \eta\left(\int_0^1 (\partial_z w)^2 \mathrm{d}z\right)\partial_z^2 w = f\right)$

**Property: the nonlinear operator preserves the co-linearity of functions $e_m$**

$$w(z,t) = E(z)^T W(t) \Rightarrow -\eta\left(\int_0^1 (\partial_z w)^2 \mathrm{d}z\right)\partial_z^2 w = \eta(W^T K^2 W) K^2 W$$

**Exact reduced order state-space representation**

$$\dot{x}(t) = A\,x(t) + B\,u(t) + \underbrace{A_3\big(x(t), x(t), x(t)\big)}_{\text{multi-linear op.}} \quad \text{and} \quad x(0) = 0_{2M\times 1},$$

$$A_3(a,b,c) = \eta\left(a^T \begin{bmatrix} K^2 & 0_{M\times M} \\ 0_{M\times M} & 0_{M\times M}\end{bmatrix} b\right)\begin{bmatrix} 0_{M\times M} & 0_{M\times M} \\ -K^2 & 0_{M\times M}\end{bmatrix} c$$

**$A_3$ is bounded** $\left(\|A_3\| = \sup_{\|x_1\|_{\mathbb{X}} = \cdots = \|x_3\|_{\mathbb{X}} = 1} \|A_3(x_1, x_2, x_3\|_{\mathbb{X}}\right)$

$$\|A_3\| \leq \frac{\eta}{2\sqrt{10}}$$

## Solution as a series expansion with respect to input $u$ (Volterra series)

**Problem:** Solve $\dot{x}(t) = A\,x(t) + B\,u(t) + A_3\big(x(t), x(t), x(t)\big)$; $x(0) = 0$

**Idea:** (1) Mark $u$ by $\varepsilon > 0$ ($u \to \varepsilon u$); (2) Expand the solution $x$ w.r.t. $\varepsilon$

$$x(t) = \sum_{m=1}^{+\infty} \varepsilon^m x_m(t) \quad (\,+\text{ inject } + \text{ isolate each order } \varepsilon^m)$$

$x_1 = x_{\mathrm{lin}}$: solution of the linearized problem (order in $\varepsilon^1$)

$$x_1(t) = \int_0^t S(t-\tau)\,B\,u(\tau)\,\mathrm{d}\tau, \quad \text{with} \quad S(t) = \exp(A\,t)1_{t>0}.$$

$x_m$ ($m \geq 2$): homogeneous order $m$ (order in $\varepsilon^m$)

$$x_m(t) = \int_0^t S(t-\tau)\,\chi_m(\tau)\,\mathrm{d}\tau$$

$$\text{with } \chi_m(\tau) = \sum_{p_1+p_2+p_3=m} A_3\big(x_{p_1}(\tau), x_{p_2}(\tau), x_{p_3}(\tau)\big)$$

## Solution as a series expansion with respect to input $u$ (Volterra series)

Problem: Solve $\dot{x}(t) = A\,x(t) + B\,u(t) + A_3\big(x(t), x(t), x(t)\big)$; $\quad x(0) = 0$

**Idea:** (1) Mark $u$ by $\varepsilon > 0$ ($u \to \varepsilon u$); (2) Expand the solution $x$ w.r.t. $\varepsilon$

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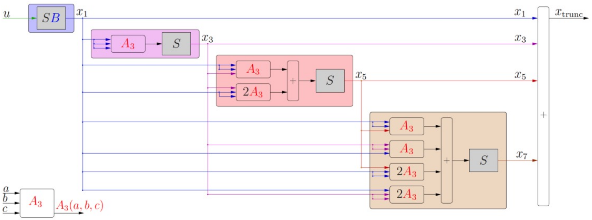


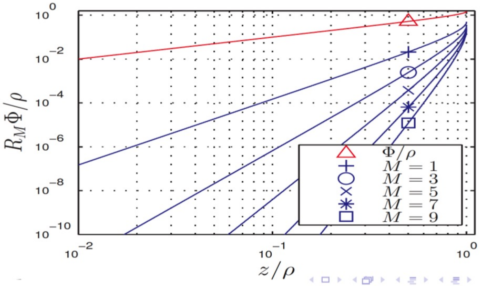
## Convergence and error bound (from Hélie & Laroche [IEEE-TAC,2010], [Automatica,2014])

**General result:** if $\|x_1\|_{\mathcal{X}} < \rho$, then

(i) $x$ is bounded (convergence in norm) $\quad \|x\|_{\mathcal{X}} < \Phi\left(\|x_1\|_{\mathcal{X}}\right)$

(ii) truncation error: idem $\quad \left\| x - \sum_{m=1}^{M} x_m \right\|_{\mathcal{X}} < R_M \Phi\left(\|x_1\|_{\mathcal{X}}\right)$

**Beam:** $\rho = \frac{2\sqrt[4]{10}}{3}\sqrt{\frac{a+b\pi^4}{\eta}}, \quad \forall z \in [0,\rho[, \quad \Phi(z) = 3\rho\cos\left(\frac{\pi + \arccos(z/\rho)}{3}\right)$

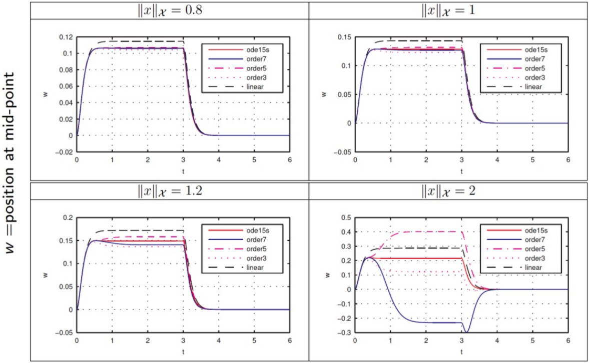


## Configuration 1: 1 mode (Duffing osc.) & excitation=pulse $u(t) = u_0 1_{[0,T]}(t)$

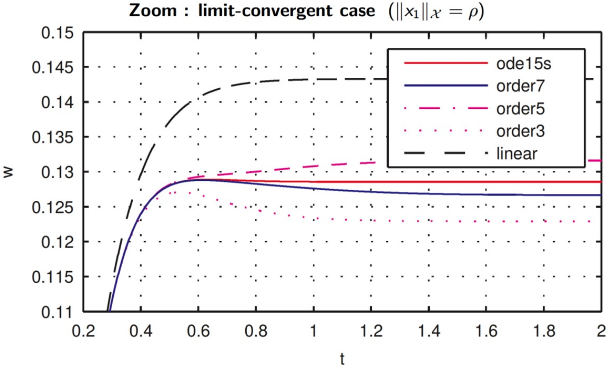
Parameters: $\rho = 1$, $a = \pi^2$, $b = 0$ (critical regime), $T = 3$s

Convergence tests: $\|x_1\|_\mathcal{X} \in \{0.8; 1; 1.2; 2\}$

Reference trajectory: solver for stiff ODEs & high sampling rate (ode15s.m)



## Configuration 1: 1 mode (Duffing osc.) & excitation=pulse $u(t) = u_0 \, 1_{[0,T]}(t)$

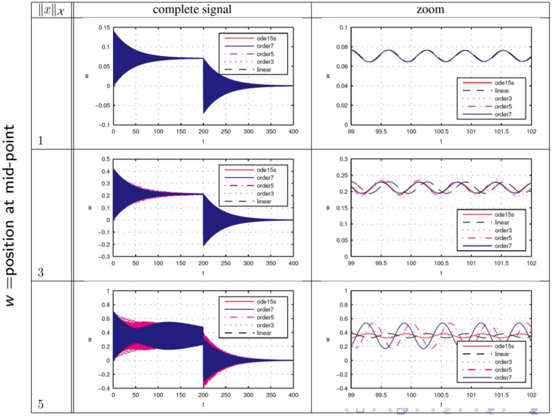


## Configuration 2: 3 modes (coupled Duffing) & excitation $u_{m\geq3}(t) = u_0\,1_{[0,T]}(t)$

Parameters: $\rho = 1$, $a = 0.02$, $b = 5\times10^{-5}$ (wooden oscillating beam), $T = 200$s

Oscillations: pitch $f_1 = 220$Hz (mode 1) for $f_{\text{sampling}} = 48000$Hz

Convergence tests: $\|x_1\|_{\mathcal{X}} \in \{1; 3; 5\}$



## Conclusion

**Contributions: pros**

- Simulation of a nonlinear model of a beam
- Regular perturbation approach (Volterra series expansion)
- Modal decomposition & set of "linear systems + static nonlinearities"
- Convergence domain and truncation error bound are characterized
- Fast convergence inside the domain: (m=1,3,5,7 are sufficient)

$\rightarrow$ **Capture distortions for any –even complex– signal shape**

**Contributions: cons = outside the convergence domain**

- (Characterized but) limited validity w.r.t. amplitude
- Increasing secular modes appear

$\rightarrow$ **do not correctly capture frequency modulations**

**Further work**

- Propose extensions to represent such modulations

$\rightarrow$ Adapted perturbation method (inspired from multiple scales ?)

# Plan

# General conclusion

Volterra series are used to represent, analyze, control and simulate some Input/Output systems which include distortions

## In this course

- **Derivation of the kernels** of a system (ODE or PDE)
- Kernels of an **inverse system (open-loop)** and a **closed-loop system (feedback)**
- **Audio applications** and **simulations**
- **Computable convergence radius** and error bound

## Other topic related to control issues: **identification**

- Hammerstein models: method based on a log-sweep
  *[Farina,AES,2000], [Novak et al.: IEEE-TIM, 2010], [Rébillat et al.: JSV, 2011]*
- Separation of orders: see e.g. Boyd *(based on amplitude)* and Bouvier *(+phase)*: https://medias.ircam.fr/x5662ab
- Group of Joannes and Maarten Schoukens (Belgium)